Inequalities

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1 Introduction

There are many classical inequalities to know for olympiad problems and handouts often give long lists of these theorems which you can commit to memory. Many can be used to prove one another. However, I find that in practice, two theorems stand out as the easiest "lens" to view inequalities with. The first is the well-known AM-GM.

Theorem 1 (AM-GM)**.** Let n be a positive integer and let $a_1, \dots, a_n$ be non-negative real numbers. Then

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n}.$$

Equality holds iff all a_i are equal.

Remark. For $n \leq 3$, the AM-GM inequality can be factored explicitly. In particular,

$$\begin{aligned} a^2 + b^2 - 2ab &= (a-b)^2, \\ a^3 + b^3 + c^3 - 3abc &= (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca) \\ &= \frac{1}{2}(a+b+c)\left((a-b)^2 + (b-c)^2 + (c-a)^2\right) \end{aligned}$$

Example 1. Let a, b, c be positive reals such that $abc = 1$. Prove that

$$a^2 + b^2 + c^2 \geq a + b + c.$$

Proof. We homogenize the inequality and seek to prove:

$$a^2 + b^2 + c^2 \geq (a+b+c)\sqrt[3]{abc}.$$

By AM-GM, we have

$$\frac{4a^2 + b^2 + c^2}{6} \geq a^{\frac{8}{6}} b^{\frac{2}{6}} c^{\frac{2}{6}} = a\sqrt[3]{abc}.$$

Similarly,

$$\frac{a^2 + 4b^2 + c^2}{6} \geq b\sqrt[3]{abc}, \quad \frac{a^2 + b^2 + 4c^2}{6} \geq c\sqrt[3]{abc}.$$

Summing up these inequalities yields the desired result. □

Example 2 (Mexico 2011)**.** Let n be a positive integer. Find all real solutions $(a_1, a_2, \dots, a_n)$ to the system:

$$\begin{gathered} a_1^2 + a_1 - 1 = a_2 \\ a_2^2 + a_2 - 1 = a_3 \\ \cdots \\ a_n^2 + a_n - 1 = a_1. \end{gathered}$$

Proof. Rearranging each equation gives

$$a_i(a_i + 1) = a_{i+1} + 1.$$

So if $a_i = -1$ for some i, we must have $a_{i+1} = -1$ and hence all the a's must be -1. This is a valid solution. Otherwise, assume $a_i \neq -1$ for all i. Taking the product, we get

$$\prod_{i=1}^{n} a_i(a_i + 1) = \prod_{i=1}^{n} (a_{i+1} + 1) \tag{1}$$

$$\implies \prod_{i=1}^{n} a_i = 1. \tag{2}$$

Now summing all n of the original equations, we get

$$a_1^2 + \cdots + a_n^2 = n.$$

By AM-GM,

$$\begin{aligned} 1 &= \frac{a_1^2 + \cdots + a_n^2}{n} \\ &\geq \sqrt[n]{a_1^2 \cdots a_n^2} \\ &= 1. \end{aligned}$$

So we must have had the equality case where $a_1^2 = a_2^2 = \cdots = a_n^2 = 1$. Since $a_i \neq -1$ in this case, we see that we must have $a_i = 1$ for all i. It is easy to check that this works.

So the real solutions are $(-1, \dots, -1)$ and $(1, \dots, 1)$. □

There is also a more general version of the AM-GM inequality with variable weights.

Theorem 2 (Weighted AM-GM)**.** Let n be a positive integer and let $a_1, \dots, a_n$ be non-negative real numbers. Let $w_1, \dots, w_n$ be non-negative real numbers with $w_1 + \cdots + w_n = 1$. Then

$$w_1 a_1 + \cdots + w_n a_n \geq a_1^{w_1} \cdots a_n^{w_n}.$$

Equality holds iff all a_i are equal.

Note that the conventional AM-GM is simply weighted AM-GM where all weights are set to $\frac{1}{n}$.

Example 3 (CMO 1995)**.** Let a, b, c be positive reals. Prove that

$$a^a b^b c^c \geq (abc)^{\frac{a+b+c}{3}}.$$

Proof. By weighted GM-HM, we have

$$\begin{aligned} a^{\frac{a}{a+b+c}} b^{\frac{b}{a+b+c}} c^{\frac{c}{a+b+c}} &\geq \left(a^{-1} \cdot \frac{a}{a+b+c} + b^{-1} \cdot \frac{b}{a+b+c} + c^{-1} \cdot \frac{c}{a+b+c}\right)^{-1} && (3) \\ &= \frac{a+b+c}{3} && (4) \\ &\geq \sqrt[3]{abc}. && (5) \end{aligned}$$

Taking to the power of $a+b+c$, we get

$$a^a b^b c^c \geq (abc)^{\frac{a+b+c}{3}}$$

as desired. □

The other useful theorem is Hölder's inequality. Let's begin by discussing Cauchy-Schwarz inequality, a special instance of Hölder's inequality.

Theorem 3 (Cauchy-Schwarz Inequality)**.** Let n be a positive integer. For any real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$, we have

$$\left(a_1^2 + \cdots + a_n^2\right)\left(b_1^2 + \cdots + b_n^2\right) \geq \left(a_1 b_1 + \cdots + a_n b_n\right)^2.$$

Equality holds iff all ratios are equal, i.e. $a_1 : b_1 = \cdots = a_n : b_n$.

Cauchy-Schwarz can be used to handle fractions.

Example 4. Show that for all positive reals a, b, c, d,

$$\frac{1}{a} + \frac{1}{b} + \frac{4}{c} + \frac{16}{d} \geq \frac{64}{a+b+c+d}.$$

Proof. By Cauchy-Schwarz, we have

$$\left(\frac{1}{a} + \frac{1}{b} + \frac{4}{c} + \frac{16}{d}\right)(a+b+c+d) \geq (1+1+2+4)^2 = 64.$$

Rearranging gives the desired inequality. □

Indeed, Cauchy-Schwarz can be written in the following form:

Theorem 4 (Titu's Lemma)**.** Let n be a positive integer. For any positive real numbers $x_1, \dots x_n$ and $y_1, \dots, y_n$, we have

$$\frac{x_1^2}{y_1} + \cdots + \frac{x_n^2}{y_n} \geq \frac{(x_1 + \cdots + x_n)^2}{y_1 + \cdots + y_n}.$$

Sometimes, a direct application of Titu's doesn't work. In these situations, you can use additional factors.

Example 5 (Nesbitt's Inequality)**.** Let a, b, c be positive reals. Prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Proof. By Cauchy-Schwarz,

$$\left(\frac{a}{b+c} + \frac{c}{a+b} + \frac{b}{c+a}\right)(a(b+c) + b(c+a) + c(a+b)) \geq (a+b+c)^2.$$

So it suffices to prove that

$$(a+b+c)^2 \geq 3(ab+bc+ca).$$

After expanding, this is equivalent to

$$a^2 + b^2 + c^2 \geq ab + bc + ca,$$

which is true because

$$\frac{a^2+b^2}{2} \geq ab, \quad \frac{b^2+c^2}{2} \geq bc, \quad \frac{c^2+a^2}{2} \geq ca.$$

□

Theorem 5 (Hölder's Inequality)**.** Let n and k be positive integers. Let $\lambda_1, \dots, \lambda_k$ be positive real numbers such that $\lambda_1 + \cdots + \lambda_k = 1$. For any k sequences of non-negative real numbers $a_1, \dots a_n; b_1, \dots, b_n; \dots; z_1, \dots, z_n$, we have

$$(a_1 + \cdots + a_n)^{\lambda_1}(b_1 + \cdots + b_n)^{\lambda_2} \cdots (z_1 + \cdots + z_n)^{\lambda_k} \geq a_1^{\lambda_1} b_1^{\lambda_2} \cdots z_1^{\lambda_k} + \cdots + a_n^{\lambda_1} b_n^{\lambda_2} \cdots z_n^{\lambda_k}.$$

Equality holds iff all ratios are equal, i.e. $a_1 : b_1 : \cdots : z_1 = \cdots = a_n : b_n : \cdots : z_n$.

Though Hölder's looks extremely intimidating in this form, it usually resembles something closer to Cauchy-Schwarz. For example, if $n = 2, k = 3$ and $\lambda_i = \frac{1}{k}$, then Hölder's gives

$$(a_1^3 + a_2^3)(b_1^3 + b_2^3)(c_1^3 + c_2^3) \geq (a_1 b_1 c_1 + a_2 b_2 c_2)^3 \quad \forall\, a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}_{\geq 0}.$$

Hölder's is excellent against radicals.

Example 6 (IMO 2001)**.** Prove that for all positive real numbers a, b, c,

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}}\geq 1.$$

Proof. There are many approaches to this problem (check out Yufei Zhao's handout for a few) but the most straightforward solution is to use Hölder's inequality.

By Hölder's, we have

$$\left(\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}}\right)^2\big(a(a^2+8bc)+b(b^2+8ca)+c(c^2+8ab)\big)\geq (a+b+c)^3.$$

Thus, it suffices to prove that

$$\begin{aligned} & (a+b+c)^3 \geq a^3+b^3+c^3+24abc \\ \iff & a^3+b^3+c^3+3\sum_{\text{sym}} a^2b+6abc \geq a^3+b^3+c^3+24abc \\ \iff & 3\sum_{\text{sym}} a^2b \geq 18abc \end{aligned}$$

which is true by AM-GM, so we are done. □

Warning. AM-GM and Hölder's are useful, but it is still important to study the other classical inequalities and techniques. For example, these two are powerless against

$$a^3+b^3+c^3+3abc \geq a^2b+ab^2+b^2c+bc^2+c^2a+ca^2 \quad \forall\, a,b,c\in\mathbb{R}_{\geq 0},$$

which is just Schur's inequality. In fact, AM-GM is one of the weakest inequalities. The reason why most olympiad inequalities are susceptible to it is because contest setters generally select for problems where knowing powerful techniques is not a large advantage. In particular, certain techniques such as the $n-1$ EV method shut down a large class of inequalities. I would recommend Thomas Mildorf's handout for reviewing inequality fundamentals.

2 Fantasy

Imagine a world where all inequalities can be solved elegantly with the right observation...

Example 7. Consider the following three problems.

(i) (Canada 2002). Let a, b, c be positive real numbers. Prove that

$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} \geq a + b + c.$$

(ii) (ToT 2021). Let $a_1, \ldots, a_n$ be positive integers. Prove that

$$\left\lfloor \frac{a_1^2}{a_2} \right\rfloor + \ldots + \left\lfloor \frac{a_n^2}{a_1} \right\rfloor \geq a_1 + \ldots + a_n.$$

(iii) (Balkan 2005). Let a, b, c be positive real numbers. Prove that

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq a + b + c + \frac{4(a-b)^2}{a+b+c}.$$

What's the common theme?

Proof. The idea is to use simple AM-GMs on each term.

(i) By AM-GM, we have

$$\frac{a^3}{bc} + b + c \geq 3a, \quad \frac{b^3}{ca} + c + a \geq 3b, \quad \frac{c^3}{ab} + a + b \geq 3c.$$

Summing yields the desired inequality.

(ii) By AM-GM, we have

$$\frac{a_i^2}{a_{i+1}} \geq 2a_i - a_{i+1}.$$

Since the right-hand side is an integer, we have the stronger inequality,

$$\left\lfloor \frac{a_i^2}{a_{i+1}} \right\rfloor \geq 2a_i - a_{i+1}.$$

Summing over all i yields the desired inequality.

(iii) We have

$$\begin{aligned}
\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} - a - b - c &= \left(\frac{a^2}{b} - 2a + b\right) + \left(\frac{b^2}{c} - 2b + c\right) + \left(\frac{c^2}{a} - 2c + a\right) \\
&= \frac{(a-b)^2}{b} + \frac{(b-c)^2}{c} + \frac{(c-a)^2}{a} \\
&\geq \frac{(|a-b| + |b-c| + |c-a|)^2}{a+b+c} \\
&\geq \frac{4(a-b)^2}{a+b+c}.
\end{aligned}$$

For the first inequality, we used Cauchy-Schwarz.

□

Example 8. Consider the following two problems.

(i) Let a, b, c be non-negative real numbers such that $ab + bc + ca = 3$. Prove that

$$\frac{1}{a^2+2} + \frac{1}{b^2+2} + \frac{1}{c^2+2} \leq 1.$$

(ii) (Mexico 2009). Let a, b, and c be positive numbers satisfying $abc = 1$. Prove that

$$\frac{1}{a^3+2} + \frac{1}{b^3+2} + \frac{1}{c^3+2} \leq 1.$$

What's the common theme?

Proof. The trick is to rearrange the fractions so that it's the right direction for Cauchy-Schwarz.

(i) Note that $\frac{1}{a^2+2} = \frac{1}{2}\left(1 - \frac{a^2}{a^2+2}\right)$ so

$$\frac{1}{a^2+2} + \frac{1}{b^2+2} + \frac{1}{c^2+2} \leq 1 \quad \iff \quad \frac{a^2}{a^2+2} + \frac{b^2}{b^2+2} + \frac{c^2}{c^2+2} \geq 1.$$

Now by Cauchy-Schwarz,

$$\begin{aligned}\frac{a^2}{a^2+2} + \frac{b^2}{b^2+2} + \frac{c^2}{c^2+2} &\geq \frac{(a+b+c)^2}{a^2+b^2+c^2+6} \\ &= \frac{(a+b+c)^2}{a^2+b^2+c^2+2ab+2bc+2ca} \\ &= 1.\end{aligned}$$

(ii) By the same trick, it suffices to prove

$$\frac{a^3}{a^3+2} + \frac{b^3}{b^3+2} + \frac{c^3}{c^3+2} \geq 1.$$

By Cauchy-Schwarz,

$$\begin{aligned}\frac{a^3}{a^3+2} + \frac{b^3}{b^3+2} + \frac{c^3}{c^3+2} &= \frac{a^3}{a^3+2abc} + \frac{b^3}{b^3+2abc} + \frac{c^3}{c^3+2abc} \\ &= \frac{a^2}{a^2+2bc} + \frac{b^2}{b^2+2ca} + \frac{c^2}{c^2+2ab} \\ &\geq \frac{(a+b+c)^2}{a^2+b^2+c^2+2ab+2bc+2ca} \\ &= 1.\end{aligned}$$

□

Example 9. Consider the following three problems.

(i) (Vietnam 1998). Let $x_1, \ldots, x_n$ be positive real numbers such that

$$\frac{1}{x_1 + 1998} + \ldots + \frac{1}{x_n + 1998} = \frac{1}{1998}.$$

Prove that

$$\frac{\sqrt[n]{x_1 \cdots x_n}}{n-1} \geq 1998.$$

(ii) (USAMO 1998). Let $a_0, a_1, \cdots, a_n$ be numbers from the interval $(0, \pi/2)$ such that

$$\tan\left(a_0 - \frac{\pi}{4}\right) + \tan\left(a_1 - \frac{\pi}{4}\right) + \cdots + \tan\left(a_n - \frac{\pi}{4}\right) \geq n - 1.$$

Prove that

$$\tan a_0 \tan a_1 \cdots \tan a_n \geq n^{n+1}.$$

(iii) Let n and k be positive integers. Show that for any positive reals $a_1, \ldots, a_n$ such that $a_1 + \ldots + a_n = 1$, we have

$$\prod_{i=1}^{n} \left(\frac{1 - a_i^k}{a_i^k}\right) \geq (n^k - 1)^n.$$

What's the common theme?

Proof. The idea is to expand 1.

(i) Define $y_i = \frac{1998}{x_i + 1998} \in \mathbb{R}_{>0}$. Then $x_i = 1998\left(\frac{1-y_i}{y_i}\right)$. With these variables, we have

$$y_1 + \cdots + y_n = 1$$

and we want to prove

$$\prod_{i=1}^{n} \left(\frac{1 - y_i}{y_i}\right) \geq (n-1)^n.$$

For any i, we have

$$\begin{aligned} 1 - y_i &= (y_1 + \cdots + y_n) - y_i \\ &= \sum_{j \neq i} y_j \\ &\geq (n-1) \prod_{j \neq i} y_j^{\frac{1}{n-1}}. \end{aligned}$$

So then

$$\begin{aligned} \prod_{i=1}^{n} \left(\frac{1 - y_i}{y_i}\right) &\geq (n-1)^n \prod_{i=1}^{n} \left(y_i^{-1} \prod_{j \neq i} y_j^{\frac{1}{n-1}}\right) \\ &= (n-1)^n \end{aligned}$$

as desired.

(ii) Define y_i as $\frac{1}{\tan a_i + 1}$. Then the condition becomes

$$y_1 + \cdots + y_n \leq 1$$

and the inequality becomes

$$\prod_{i=0}^{n} \left(\frac{1 - y_i}{y_i} \right) \geq n^{n+1}.$$

This is essentially the same inequality as the one we reduced to in (i), and the same solution works.

(iii) For any i, we have

$$\begin{aligned} 1 - a_i^k &= (a_1 + \cdots + a_n)^k - a_i^k \\ &= \sum_{\substack{j_1, \ldots, j_k \\ (j_1, \ldots, j_k) \neq (i, \ldots, i)}} a_{j_1} \cdots a_{j_k} \\ &\geq (n^k - 1) \left(\prod_{\substack{j_1, \ldots, j_k \\ (j_1, \ldots, j_k) \neq (i, \ldots, i)}} a_{j_1} \cdots a_{j_k} \right)^{\frac{1}{n^k - 1}} \\ &= (n^k - 1) a_i^{\frac{kn^{k-1} - k}{n^k - 1}} \prod_{j \neq i} a_j^{\frac{kn^{k-1}}{n^k - 1}}. \end{aligned}$$

For example, if $n = 2, k = 2$, we have

$$\begin{aligned} 1 - a_1^2 &= (a_1 + a_2)^2 - a_1^2 \\ &= a_1 a_2 + a_1 a_2 + a_2^2 \\ &= 3 \sqrt[3]{a_1^2 a_2^4}. \end{aligned}$$

So we have

$$\begin{aligned} \prod_{i=1}^{n} \left(\frac{1 - a_i^k}{a_i^k} \right) &\geq (n^k - 1)^n \prod_{i=1}^{n} \left(a_i^{\frac{kn^{k-1} - k}{n^k - 1} - k} \prod_{j \neq i} a_j^{\frac{kn^{k-1}}{n^k - 1}} \right) \\ &= (n^k - 1)^n \prod_{i=1}^{n} a_i^{\frac{kn^{k-1} - k}{n^k - 1} - k + \frac{kn^{k-1}(n-1)}{n^k - 1}} \\ &= (n^k - 1)^n \end{aligned}$$

as desired.

□

Example 10. Consider the following three problems.

(i) (USAJMO 2014). Let a, b, c be real numbers greater than or equal to 1. Prove that

$$\min\left(\frac{10a^2-5a+1}{b^2-5b+10}, \frac{10b^2-5b+1}{c^2-5c+10}, \frac{10c^2-5c+1}{a^2-5a+10}\right) \le abc.$$

(ii) (Serbia 2022). Let a, b and c be positive real numbers and $a^3+b^3+c^3=3$. Prove

$$\frac{1}{3-2a}+\frac{1}{3-2b}+\frac{1}{3-2c} \ge 3.$$

(iii) (USAMO 2004). Let $a,b,c>0$. Prove that

$$(a^5-a^2+3)(b^5-b^2+3)(c^5-c^2+3) \ge (a+b+c)^3.$$

What's the common theme?

Proof. The key for these problems is to make 1-variable polynomial bounds. □

(i) First note that for any $x \ge 1$, $x^2-5x+10>0$. Then we have

$$\begin{aligned} &(x-1)^5 \ge 0 \\ \implies & x^5-5x^4+10x^3 \ge 10x^2-5x+1 \\ \implies & x^3 \ge \frac{10x^2-5x+1}{x^2-5x+10}. \end{aligned}$$

Let's now bound the minimum by their geometric mean:

$$\begin{aligned} \min\left(\frac{10a^2-5a+1}{b^2-5b+10}, \frac{10b^2-5b+1}{c^2-5c+10}, \frac{10c^2-5c+1}{a^2-5a+10}\right) &\le \sqrt[3]{\prod_{\text{cyc}} \frac{10a^2-5a+1}{b^2-5b+10}} \\ &= \sqrt[3]{\prod_{\text{cyc}} \frac{10a^2-5a+1}{a^2-5a+10}} \\ &\le abc \end{aligned}$$

as desired.

(ii) Note that $a,b,c<\frac{3}{2}$ as $\left(\frac{3}{2}\right)^3>3$. If we could prove that

$$\frac{1}{3-2x} \ge ux^3+v$$

for $x \in (0, \frac{3}{2})$ and some suitable constants $u,v \in \mathbb{R}$, that would essentially solve the problem. The trick to deriving this inequality is to set up function $P(x)$ so that

$$\frac{1}{3-2x} \ge ux^3+v \quad \iff \quad P(x) := 2ux^4-3ux^3+2vx-3v+1 \ge 0.$$

Based on the equality case, we need $x = 1$ to be a root of P. Since P must be non-negative, we also need a double-root at $x = 1$. This corresponds to

$$\begin{aligned} P(1) = 0 &\implies u + v = 1 \\ P'(1) = 0 &\implies \left(8ux^3 - 9ux^2 + 2v\right)|_{x=1} = 0 \\ &\implies u - 2v = 0 \end{aligned}$$

Solving gives $u = \frac{2}{3}, v = \frac{1}{3}$.

Though this gives us a candidate set of coefficients, we still need to verify that for any $x \in (0, \frac{3}{2})$, we have

$$\frac{1}{3-2x} \geq \frac{2x^3+1}{3}.$$

This is indeed true as it rearranges into $x(x-1)^2(4x+2) \geq 0$.

So

$$\frac{1}{3-2a} + \frac{1}{3-2b} + \frac{1}{3-2c} \geq \frac{2\left(a^3+b^3+c^3\right)+3}{3} = 3.$$

(iii) For any $x > 0$, we have

$$x^5 - x^2 + 3 \geq x^3 + 2$$

as it rearranges into

$$(x^3-1)(x^2-1) = (x-1)^2(x+1)(x^2+x+1) \geq 0.$$

So

$$\begin{aligned} (a^5-a^2+3)(b^5-b^2+3)(c^5-c^2+3) &\geq (a^3+2)(b^3+2)(c^3+2) \\ &= (a^3+1+1)(1+b^3+1)(1+1+c^3) \\ &\geq (a+b+c)^3 \end{aligned}$$

by Hölder's inequality.

Example 11. Consider the following three problems.

(i) Let a, b, c be reals with $a+b+c=1$ and $a, b, c \geq -\frac{3}{4}$. Prove that

$$\frac{a}{a^2+1}+\frac{b}{b^2+1}+\frac{c}{c^2+1} \leq \frac{9}{10}.$$

(ii) (CGMO 2007). Let n be an integer greater than 3, and let $a_1, a_2, \cdots, a_n$ be non-negative real numbers with $a_1+a_2+\cdots+a_n=2$. Determine the minimum value of

$$\frac{a_1}{a_2^2+1}+\frac{a_2}{a_3^2+1}+\cdots+\frac{a_n}{a_1^2+1}.$$

(iii) (USAMO 2017). Find the minimum possible value of

$$\frac{a}{b^3+4}+\frac{b}{c^3+4}+\frac{c}{d^3+4}+\frac{d}{a^3+4}$$

given that a, b, c, d are nonnegative real numbers such that $a+b+c+d=4$.

What's the common theme?

Proof. We use the tangent line trick for these problems.

(i) Consider $f(x)=\frac{x}{x^2+1}$. We can compute its derivative, $f'(x)=\frac{1\cdot(x^2+1)-x\cdot(2x)}{(2x+1)^2}$. At $a=\frac{1}{3}$, we have $f(a)=\frac{3}{10}$ and $f'(a)=\frac{18}{25}$. So the tangent line at $a=\frac{1}{3}$ is

$$g(x)=f'(a)(x-a)+f(a)=\frac{18}{25}x+\frac{3}{50}.$$

We claim that for $x \geq -\frac{3}{4}$, $g(x) \geq f(x)$. Indeed,

$$\frac{18}{25}x+\frac{3}{50} \geq \frac{x}{x^2+1} \tag{6}$$

$$\iff 36x^3+3x^2-14x+3 \geq 0 \tag{7}$$

$$\iff \quad (3x-1)^2(4x+3) \geq 0 \tag{8}$$

so the claim is proved. Now

$$\frac{a}{a^2+1}+\frac{b}{b^2+1}+\frac{c}{c^2+1} \leq \frac{18}{25}(a+b+c)+\frac{9}{50}=\frac{9}{10}.$$

(ii) We claim that the minimum value is $\frac{3}{2}$, which is achieved when $a_1=1, a_2=1$ and the rest are 0.

Note that for any $x \geq 0$, we have

$$\frac{1}{1+x^2} \geq 1-\frac{x}{2},$$

as it rearranges into $x(x-1)^2 \geq 0$. So

$$\frac{a_1}{a_2^2+1}+\frac{a_2}{a_3^2+1}+\cdots+\frac{a_n}{a_1^2+1} \geq \sum_{i=1}^{n} a_i\left(1-\frac{a_{i+1}}{2}\right)=2-\frac{1}{2}\sum_{i=1}^{n} a_i a_{i+1}.$$

It now suffices to prove that

$$\sum_{i=1}^{n} a_i a_{i+1} \leq 1.$$

The finish is actually quite tricky from here, and relies on $n \geq 4$. Since the sum is cyclic, we can say without loss of generality that a_2 is the largest among the n numbers. Say that among $\{a_4, \ldots, a_n\}$, a_k is the largest. Then

$$\begin{aligned}
a_1a_2 + a_2a_3 + a_3a_4 + \ldots + a_na_1 &\leq (a_1a_2 + a_2a_3) + \left(\sum_{i=3}^{k-1} a_ia_{i+1}\right) + \left(\sum_{i=k}^{n} a_ia_{i+1}\right) \\
&\leq (a_1a_2 + a_2a_3) + \left(\sum_{i=3}^{k-1} a_ia_k\right) + \left(\sum_{i=k}^{n} a_ka_{i+1}\right) \\
&\leq a_2(2 - a_2 - a_k) + a_k(2 - a_2 - a_k) \\
&\leq \left(\frac{(2 - a_2 - a_k) + (a_2 + a_k)}{2}\right)^2 \\
&= 1.
\end{aligned}$$

(iii) We claim that the minimum value is $\frac{2}{3}$, which can be achieved at $(2, 2, 0, 0)$. Using the tangent line trick focusing on equality at $x = 2$, we see that

$$\frac{1}{x^3 + 4} \geq \frac{1}{4} - \frac{x}{12} \qquad \iff \qquad x(x+1)(x-2)^2 \geq 0.$$

So

$$\begin{aligned}
\frac{a}{b^3 + 4} + \frac{b}{c^3 + 4} + \frac{c}{d^3 + 4} + \frac{d}{a^3 + 4} &\geq \frac{1}{4}(a + b + c + d) - \frac{1}{12}(ab + bc + cd + da) \\
&= 1 - \frac{1}{12}(a + c)(b + d) \\
&\geq 1 - \frac{1}{12}\left(\frac{(a + c) + (b + d)}{2}\right)^2 \\
&= \frac{2}{3}.
\end{aligned}$$

□

Remark. In (ii) and (iii), the key bounds can also be derived without explicitly thinking about the tangent line trick. For example, in (iii), we could have instead used AM-GM after rearranging the fraction:

$$\begin{aligned}
\frac{a}{b^3 + 4} &= \frac{a}{4}\left(1 - \frac{b^3}{b^3 + 4}\right) \\
&= \frac{a}{4}\left(1 - \frac{b^3}{\frac{b^3}{2} + \frac{b^3}{2} + 4}\right) \\
&\geq \frac{a}{4}\left(1 - \frac{b^3}{3b^2}\right) \\
&= \frac{a}{4} - \frac{ab}{12}.
\end{aligned}$$

Example 12. Consider the following three problems.

(i) (Taiwan 2014). Prove that for positive reals a, b, c we have

$$3(a+b+c) \geq 8\sqrt[3]{abc} + \sqrt[3]{\frac{a^3+b^3+c^3}{3}}.$$

(ii) Let $x_1, \ldots, x_n$ be non-negative real numbers. Prove that

$$x_1 + x_2 + \cdots + x_n \geq (n-1)\sqrt[n]{x_1 \cdots x_n} + \sqrt{\frac{x_1^2 + \cdots + x_n^2}{n}}.$$

(iii) Show that for all non-negative $a_1, \ldots, a_n$,

$$\frac{a_1 + \sqrt{a_1 a_2} + \cdots + \sqrt[n]{a_1 \cdots a_n}}{n} \leq \sqrt[n]{a_1 \cdot \frac{a_1+a_2}{2} \cdots \frac{a_1 + \cdots + a_n}{n}}.$$

What's the common theme?

Proof. The key is to set up for Hölder's.

(i) Taking the cube, we have

$$\begin{aligned}(3(a+b+c))^3 &= 27\left(a^3+b^3+c^3+3\sum_{\text{sym}} a^2b + 6abc\right)\\ &\geq 27\left(a^3+b^3+c^3+24abc\right)\\ &= (8+1)(8+1)\left(8abc + \frac{a^3+b^3+c^3}{3}\right)\\ &\geq \left(8\sqrt[3]{abc} + \sqrt[3]{\frac{a^3+b^3+c^3}{3}}\right)^3\end{aligned}$$

as desired.

(ii) Squaring gives us

$$\begin{aligned}(x_1+\cdots+x_n)^2 &= \left(x_1^2+\cdots+x_n^2\right) + \sum_{i\neq j} x_i x_j\\ &\geq \left(x_1^2+\cdots+x_n^2\right) + n(n-1)\sqrt[n]{x_1^2 \cdots x_n^2}\\ &= ((n-1)+1)\left((n-1)\sqrt[n]{x_1^2\cdots x_n^2} + \frac{x_1^2+\cdots+x_n^2}{n}\right)\\ &\geq (n-1)\sqrt[n]{x_1\cdots x_n} + \sqrt{\frac{x_1^2+\cdots+x_n^2}{n}}\end{aligned}$$

as desired.

(iii) Let g_k denote the geometric mean of $a_1, \dots, a_k$ and define $g_0 = 1$. We proceed by induction. For $n = 1$, the statement is obvious. Now assume the statement holds for $n-1$. Then

$$\sqrt[n]{a_1 \cdot \frac{a_1+a_2}{2} \cdots \frac{a_1+\cdots+a_n}{n}} \geq \sqrt[n]{\left(\frac{g_1+g_2+\cdots+g_{n-1}}{n-1}\right)^{n-1} \cdot \left(\frac{a_1+\cdots+a_n}{n}\right)}.$$

We will apply Hölder's in a clever manner. Consider $\frac{g_1+g_2+\cdots+g_{n-1}}{n-1}$. We redistribute this into n terms as follows:

$$\begin{aligned}
\frac{g_1+g_2+\cdots+g_{n-1}}{n-1} &= \frac{n}{n(n-1)}g_1 + \frac{n}{n(n-1)}g_2 + \cdots + \frac{n}{n(n-1)}g_{n-1} \\
&= \left(\frac{n-1}{n(n-1)}g_1\right) + \left(\frac{1}{n(n-1)}g_1 + \frac{n-2}{n(n-1)}g_2\right) + \cdots \\
&\quad + \left(\frac{n-2}{n(n-1)}g_{n-2} + \frac{1}{n(n-1)}g_{n-1}\right) + \left(\frac{n-1}{n(n-1)}g_{n-1}\right) \\
&= \frac{1}{n}\sum_{i=0}^{n-1}\left(\frac{i}{n-1}g_i + \frac{n-1-i}{n-1}g_{i+1}\right) \\
&\geq \frac{1}{n}\sum_{i=0}^{n-1}\left(g_i^i g_{i+1}^{n-1-i}\right)^{\frac{1}{n-1}} \\
&= \frac{1}{n}\sum_{i=0}^{n-1}\left(a_{i+1}^{-1} g_{i+1}^{i+1} g_{i+1}^{n-1-i}\right)^{\frac{1}{n-1}} \\
&= \frac{1}{n}\sum_{i=0}^{n-1}\left(a_{i+1}^{-1} g_{i+1}^{n}\right)^{\frac{1}{n-1}}.
\end{aligned}$$

We can now use Hölder's:

$$\begin{aligned}
\sqrt[n]{a_1 \cdot \frac{a_1+a_2}{2} \cdots \frac{a_1+\ldots+a_n}{n}} &\geq \sqrt[n]{\left(\frac{g_1+g_2+\cdots+g_{n-1}}{n-1}\right)^{n-1} \cdot \left(\frac{a_1+\ldots+a_n}{n}\right)} \\
&\geq \sqrt[n]{\left(\frac{1}{n}\sum_{i=0}^{n-1}\left(a_{i+1}^{-1}g_{i+1}^{n}\right)^{\frac{1}{n-1}}\right)^{n-1} \cdot \left(\frac{a_1+\ldots+a_n}{n}\right)} \\
&\geq \frac{1}{n}\sum_{i=1}^{n} g_i
\end{aligned}$$

as desired.

□

3 Reality

...now open your eyes.

Example 13 (IMO 2020)**.** The real numbers a, b, c, d are such that $a \geq b \geq c \geq d > 0$ and $a + b + c + d = 1$. Prove that

$$(a + 2b + 3c + 4d)a^a b^b c^c d^d < 1.$$

Proof. By weighted AM-GM, we have $a^a b^b c^c d^d \leq a^2 + b^2 + c^2 + d^2$. After homogenizing, it suffices to prove that

$$\begin{array}{rrcr} & (a+2b+3c+4d)(a^2+b^2+c^2+d^2) & < & (a+b+c+d)^3 \\ & a^3+2b^3+3c^3+4d^3 & & a^3+b^3+c^3+d^3 \\ & +ab^2+ac^2+ad^2 & & +3ab^2+3ac^2+3ad^2 \\ \iff & +2a^2b+2bc^2+2bd^2 & < & +3a^2b+3bc^2+3bd^2 \\ & +3a^2c+3b^2c+3cd^2 & & +3a^2c+3b^2c+3cd^2 \\ & +4a^2d+4b^2d+4c^2d & & +3a^2d+3b^2d+3c^2d \\ & & & +6bcd+6acd+6abd+6abc \\ \iff & b^3+2c^3+3d^3+a^2d+b^2d+c^2d & < & 2ab^2+2ac^2+2ad^2+a^2b+bc^2+bd^2 \\ & & & +6(bcd+acd+abd+abc) \end{array}$$

However, from $a \geq b \geq c \geq d > 0$, we have

$$\begin{aligned} b^3 &\leq ab^2 \\ 2c^3 &\leq 2ac^2 \\ 3d^3 &\leq 3bcd \\ a^2d &\leq a^2b \\ b^2d &\leq abd \\ c^2d &\leq acd \end{aligned}$$

so summing gives us the desired inequality. □

Remark. This is a truly terrible problem. The moral of this is that you shouldn't always try to look for a nice solution.

Example 14 (ISL 2016)**.** Let a, b, c be positive real numbers such that $\min(ab, bc, ca) \geq 1$. Prove that

$$\sqrt[3]{(a^2+1)(b^2+1)(c^2+1)} \leq \left(\frac{a+b+c}{3}\right)^2 + 1.$$

Proof. The idea is to smooth two variables. We will prove that if $xy \geq 1$, then

$$(x^2+1)(y^2+1) \leq \left(\left(\frac{x+y}{2}\right)^2 + 1\right)^2.$$

Expanding yields

$$\begin{aligned} & \left(\left(\frac{x+y}{2}\right)^2+1\right)^2 \geq (x^2+1)(y^2+1) \\ \iff & (x+y)^4+8(x+y)^2 \geq 16x^2y^2+16x^2+16y^2 \\ \iff & x^4+4x^3y-10x^2y^2+4xy^3+y^4-8(x-y)^2 \geq 0 \\ \iff & (x-y)^2(x^2+6xy+y^2-8) \geq 0 \end{aligned}$$

which is true since $x^2+6xy+y^2 \geq 8xy \geq 8$.

Furthermore,

$$\min\left(\left(\frac{a+b}{2}\right)^2, \left(\frac{a+b}{2}\right)c, \left(\frac{a+b}{2}\right)c\right) \geq \min(ab, bc, ca) \geq 1.$$

Now WLOG $a \geq b, c$ and let $d = \frac{a+b+c}{3}$. We can check that

$$ad \geq 1, \quad bc \geq 1, \quad \left(\frac{a+d}{2}\right)\left(\frac{b+c}{2}\right) \geq 1$$

and so we can apply our lemma on each of these pairs. Doing so yields

$$\begin{aligned} \left(a^2+1\right)\left(b^2+1\right)\left(c^2+1\right)\left(d^2+1\right) &\leq \left(\left(\frac{a+d}{2}\right)^2+1\right)^2\left(\left(\frac{b+c}{2}\right)^2+1\right)^2 \\ &\leq \left(\left(\frac{a+b+c+d}{4}\right)^2+1\right)^4 \\ &= \left(d^2+1\right)^4. \end{aligned}$$

Hence,

$$\sqrt[3]{(a^2+1)(b^2+1)(c^2+1)} \leq \left(\frac{a+b+c}{3}\right)^2+1$$

as desired. □

Example 15 (IMO 2021)**.** Show that the inequality

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i-x_j|} \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i+x_j|}$$

holds for all real numbers $x_1, \ldots x_n$.

Proof. The key is to note that the left-hand side does not change upon adding a constant. For any $0 < \epsilon \leq \frac{1}{2}\min_{i,j}|x_i+x_j|$, consider replacing $(x_1, \ldots, x_n)$ with

$$\begin{aligned} (x_1, \ldots, x_n) &\to (x_1+\epsilon, \ldots, x_n+\epsilon), \\ (x_1, \ldots, x_n) &\to (x_1-\epsilon, \ldots, x_n-\epsilon). \end{aligned}$$

Clearly, the left-hand-side is unaffected. We claim that one of these replacements will cause the right-hand-side to decrease. Indeed, $\sqrt{|x|}$ is strictly concave for $x \in (0, \infty)$ so

$$\sum_{i,j} \sqrt{|x_i + x_j - 2\epsilon|} + \sum_{i,j} \sqrt{|x_i + x_j + 2\epsilon|} < 2 \sum_{i,j} \sqrt{|x_i + x_j|}$$

and so the claim is proved. Hence, we can choose $\epsilon = \frac{1}{2} \min_{i,j} |x_i + x_j|$ to force some $x_i + x_j = 0$ if such a pair doesn't already exist. Without loss of generality, say that this is $x_{n-1} = t, x_n = -t$. We have

$$\begin{aligned}
& \sum_{i=1}^{n} \sum_{j=1}^{n} \sqrt{|x_i - x_j|} \leqslant \sum_{i=1}^{n} \sum_{j=1}^{n} \sqrt{|x_i + x_j|} \\
\iff \quad & 2\sqrt{|t|} + \sum_{i=1}^{n-2} \left(\sqrt{|x_i - t|} + \sqrt{|x_i + t|} \right) + \sum_{i=1}^{n-2} \sum_{j=1}^{n-2} \sqrt{|x_i - x_j|} \\
& \leqslant 2\sqrt{|t|} + \sum_{i=1}^{n-2} \left(\sqrt{|x_i - t|} + \sqrt{|x_i + t|} \right) + \sum_{i=1}^{n-2} \sum_{j=1}^{n-2} \sqrt{|x_i + x_j|} \\
\iff \quad & \sum_{i=1}^{n-2} \sum_{j=1}^{n-2} \sqrt{|x_i - x_j|} \leqslant \sum_{i=1}^{n-2} \sum_{j=1}^{n-2} \sqrt{|x_i + x_j|}.
\end{aligned}$$

We can now induct down. It remains to prove $n = 0$ and $n = 1$, both of which are trivial. So we are done. $\square$

4 Problems

A1. (CMO 2012). Let x, y and z be positive real numbers. Show that

$$x^2 + xy^2 + xyz^2 \geq 4xyz - 4.$$

A2. (CMO 2021). Let $n \geq 2$ be some fixed positive integer and suppose that $a_1, a_2, \ldots, a_n$ are positive real numbers satisfying $a_1 + a_2 + \cdots + a_n = 2^n - 1$.

Find the minimum possible value of

$$\frac{a_1}{1} + \frac{a_2}{1+a_1} + \frac{a_3}{1+a_1+a_2} + \cdots + \frac{a_n}{1+a_1+a_2+\cdots+a_{n-1}}.$$

A3. Let $P(x)$ be a polynomial with positive coefficients. Prove that for any $x \neq 0$,

$$P(x)P(x^{-1}) \geq P(1)^2.$$

A4. (CMO 2014). Let $a_1, a_2, \ldots, a_n$ be positive real numbers whose product is 1. Show that the sum

$$\tfrac{a_1}{1+a_1} + \tfrac{a_2}{(1+a_1)(1+a_2)} + \tfrac{a_3}{(1+a_1)(1+a_2)(1+a_3)} + \cdots + \tfrac{a_n}{(1+a_1)(1+a_2)\cdots(1+a_n)}$$

is greater than or equal to $\frac{2^n-1}{2^n}$.

A5. Let a, b be positive real numbers such that $a + b = 1$. Prove that

$$\frac{1}{ab} + \frac{3}{a^2+b^2} \geq 5 + 2\sqrt{6}.$$

A6. Let x, y be positive real numbers such that $x + y = 1$. Prove that

$$x^x \cdot y^y + x^y \cdot y^x \leq 1.$$

A7. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{a+b+c}{3} \geq \frac{a}{a^2b+2} + \frac{b}{b^2c+2} + \frac{c}{c^2a+2}.$$

A8. (ELMOSL 2025). Elmo writes positive real numbers on a $n \times n$ board such that $x_{i,j}$ is the number on the i-th row and j-th column. The sum of numbers in a-th row is R_a, and the sum of numbers in b-th column is C_b. It is known that

$$\sum_{a=1}^{n} \frac{x_{a,k}}{R_a} = 1$$

for each $k = 1, 2, \ldots, n$. Prove that $R_1R_2\cdots R_n \leq C_1C_2\cdots C_n$.

A9. (CMO 2017). For pairwise distinct nonnegative reals a, b, c, prove that

$$\frac{a^2}{(b-c)^2} + \frac{b^2}{(c-a)^2} + \frac{c^2}{(b-a)^2} > 2.$$

A10. (IMO 1995). Let a, b, c be positive reals with $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

A11. (IMO 2000). Let a, b, c be positive real numbers so that $abc = 1$. Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

A12. (IMO 2023). Let $x_1, x_2, \ldots, x_{2023}$ be pairwise different positive real numbers such that

$$a_n = \sqrt{(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right)}$$

is an integer for every $n = 1, 2, \ldots, 2023$. Prove that $a_{2023} \geqslant 3034$.

A13. (ELMOSL 2013). Prove that for all positive reals a, b, c,

$$\frac{1}{a + \frac{1}{b} + 1} + \frac{1}{b + \frac{1}{c} + 1} + \frac{1}{c + \frac{1}{a} + 1} \geq \frac{3}{\sqrt[3]{abc} + \frac{1}{\sqrt[3]{abc}} + 1}.$$

B1. (IMO 2012). Let $n \geq 3$ be an integer, and let $a_2, a_3, \ldots, a_n$ be positive real numbers such that $a_2 a_3 \cdots a_n = 1$. Prove that

$$(1 + a_2)^2(1 + a_3)^3 \cdots (1 + a_n)^n > n^n.$$

B2. (USAMO 2002). Let ABC be a triangle such that

$$\left(\cot\frac{A}{2}\right)^2 + \left(2\cot\frac{B}{2}\right)^2 + \left(3\cot\frac{C}{2}\right)^2 = \left(\frac{6s}{7r}\right)^2,$$

where s and r denote its semiperimeter and its inradius, respectively. Prove that triangle ABC is similar to a triangle T whose side lengths are all positive integers with no common divisors and determine these integers.

B3. (ISL 2020). Suppose that a, b, c, d are positive real numbers satisfying $(a+c)(b+d) = ac+bd$. Find the smallest possible value of

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}.$$

B4. (USAMO 2013). Find all real numbers $x, y, z \geq 1$ satisfying

$$\min(\sqrt{x + xyz}, \sqrt{y + xyz}, \sqrt{z + xyz}) = \sqrt{x - 1} + \sqrt{y - 1} + \sqrt{z - 1}.$$

B5. (ISL 2001). Let $x_1, x_2, \ldots, x_n$ be arbitrary real numbers. Prove the inequality

$$\frac{x_1}{1 + x_1^2} + \frac{x_2}{1 + x_1^2 + x_2^2} + \cdots + \frac{x_n}{1 + x_1^2 + \cdots + x_n^2} < \sqrt{n}.$$

B6. (ISL 2007). Let n be a positive integer, and let x and y be a positive real number such that $x^n + y^n = 1$. Prove that

$$\left(\sum_{k=1}^{n} \frac{1 + x^{2k}}{1 + x^{4k}}\right) \cdot \left(\sum_{k=1}^{n} \frac{1 + y^{2k}}{1 + y^{4k}}\right) < \frac{1}{(1 - x) \cdot (1 - y)}.$$

B7. Find the minimum c such that the following inequality is true for all positive numbers x, y, z:

$$\frac{x^3}{x^3+y^2z}+\frac{y^3}{y^3+z^2x}+\frac{z^3}{z^3+x^2y}\leq c.$$

B8. (Mexico 2020). Let $n \geq 2$ be a positive integer. Let $x_1, x_2, \ldots, x_n$ be non-zero real numbers satisfying the equation

$$\left(x_1+\frac{1}{x_2}\right)\left(x_2+\frac{1}{x_3}\right)\ldots\left(x_n+\frac{1}{x_1}\right)=\left(x_1^2+\frac{1}{x_2^2}\right)\left(x_2^2+\frac{1}{x_3^2}\right)\ldots\left(x_n^2+\frac{1}{x_1^2}\right).$$

Find all possible values of $x_1, x_2, \ldots, x_n$.

B9. (IMO 2018). Find all integers $n \geq 3$ for which there exist real numbers $a_1, a_2, \ldots a_{n+2}$ satisfying $a_{n+1} = a_1$, $a_{n+2} = a_2$ and

$$a_i a_{i+1} + 1 = a_{i+2},$$

for $i = 1, 2, \ldots, n$.

B10. (USAMO 2021). Let $n \geq 4$ be an integer. Find all positive real solutions to the following system of $2n$ equations:

$$\begin{aligned}
a_1 &= \frac{1}{a_{2n}}+\frac{1}{a_2}, & a_2 &= a_1+a_3,\\
a_3 &= \frac{1}{a_2}+\frac{1}{a_4}, & a_4 &= a_3+a_5,\\
a_5 &= \frac{1}{a_4}+\frac{1}{a_6}, & a_6 &= a_5+a_7\\
&\vdots & &\vdots\\
a_{2n-1} &= \frac{1}{a_{2n-2}}+\frac{1}{a_{2n}}, & a_{2n} &= a_{2n-1}+a_1
\end{aligned}$$

B11. (ISL 2021). Let $n \geq 2$ be an integer and let $a_1, a_2, \ldots, a_n$ be positive real numbers with sum 1. Prove that

$$\sum_{k=1}^{n}\frac{a_k}{1-a_k}(a_1+a_2+\cdots+a_{k-1})^2<\frac{1}{3}.$$

B12. (IMO 2006). Determine the least real number M such that the inequality

$$|ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2)|\leq M(a^2+b^2+c^2)^2$$

holds for all real numbers a, b and c.

B13. (APMO 2022). Let a, b, c, d be real numbers such that $a^2+b^2+c^2+d^2=1$. Determine the minimum value of $(a-b)(b-c)(c-d)(d-a)$ and determine all values of (a, b, c, d) such that the minimum value is achieved.

C1. (APMO 2024). Let n be a positive integer and let $a_1, a_2, \ldots, a_n$ be positive reals. Show that

$$\sum_{i=1}^{n}\frac{1}{2^i}\left(\frac{2}{1+a_i}\right)^{2^i}\geq\frac{2}{1+a_1a_2\ldots a_n}-\frac{1}{2^n}.$$

C2. (China 2018). Let A_1, A_2, $\cdots$, A_m be m subsets of a set of size n. Prove that

$$\sum_{i=1}^{m}\sum_{j=1}^{m}|A_i|\cdot|A_i\cap A_j|\geq\frac{1}{mn}\left(\sum_{i=1}^{m}|A_i|\right)^3.$$

C3. (ISL 2018). Find the maximal value of

$$S=\sqrt[3]{\frac{a}{b+7}}+\sqrt[3]{\frac{b}{c+7}}+\sqrt[3]{\frac{c}{d+7}}+\sqrt[3]{\frac{d}{a+7}},$$

where a, b, c, d are nonnegative real numbers which satisfy $a+b+c+d=100$.

C4. (ISL 2004). Let $a_1, a_2, \ldots, a_n$ be positive real numbers, $n>1$. Denote by g_n their geometric mean, and by $A_1, A_2, \ldots, A_n$ the sequence of arithmetic means defined by

$$A_k=\frac{a_1+a_2+\cdots+a_k}{k},\qquad k=1,2,\ldots,n.$$

Let G_n be the geometric mean of $A_1, A_2, \ldots, A_n$. Prove the inequality

$$n\sqrt[n]{\frac{G_n}{A_n}}+\frac{g_n}{G_n}\leq n+1$$

and establish the cases of equality.

C5. Let a, b, c, d be non-negative real numbers such that $a+b+c+d=6$. Prove that

$$(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)\leq 27.$$